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Journal of Approximation Theory 129 (2004) 1–27
Journal of Approximation Theory
http://www.elsevier.com/locate/jat

Approximation numbers in function spaces and the distribution of eigenvalues of some fractal elliptic operators

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Received 26 May 2003; accepted in revised form 19 May 2004

Communicated by Peter Oswald

Abstract

The aim of this paper is threefold. Firstly, we deal with approximation numbers of compact embeddings

$$B^s_{pp}(\mathbb{R}^n) \hookrightarrow L_p(\mu), \quad s>0, \ 1<p<\infty,$$

where μ is an (isotropic) Radon measure in $\mathbb{R}^n$. Secondly, we apply the outcome to study the distribution of the eigenvalues of fractal elliptic operators

$$B_s = (id - \Delta)^{-s} \circ \mu, \quad s>0.$$

Thirdly, we wish to demonstrate that the theory of subatomic wavelet frames in function spaces according to (Studia Math. 154 (2003) 59) is an efficient tool to handle problems of this and related type.

MSC: 46E35; 42B35; 28A80; 42C40; 35P15

Keywords: Function spaces; Radon measures; Fractals; Distribution of eigenvalues; Wavelet frames

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doi:10.1016/j.jat.2004.05.003

1. Introduction

Let μ be a positive Radon measure in $\mathbb{R}^n$ with

$$\Gamma = \operatorname{supp}\mu \quad \text{compact}, \quad 0<\mu(\mathbb{R}^n)<\infty, \quad |\Gamma| = 0, \tag{1.1}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. Let

$$B^s_p(\mathbb{R}^n) = B^s_{pp}(\mathbb{R}^n), \quad 1<p<\infty, \quad 0<s\leqslant\frac{n}{p},$$

be special classical Besov spaces. The aim of this paper is threefold.

First, we ask for existence and properties of the trace operator tr_μ,

$$tr_\mu : B^s_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma,\mu). \tag{1.2}$$

If μ is isotropic then one gets definitive answers. Recall that the above Radon measure μ is called *isotropic* if there is a function h, defined, non-negative, continuous, and strictly increasing on the interval $[0,1]$ with $h(0)=0$, $h(1)=1$, such that

$$\mu(B(\gamma,r)) \sim h(r), \quad \gamma\in\Gamma, \ 0<r<1,$$

where $B(\gamma,r)$ is a ball in $\mathbb{R}^n$ centred at γ and of radius r. As for the meaning of $\sim$ we refer to (2.1). It comes out (Theorem 1) that tr_μ according to (1.2) exists (as a continuous map) if, and only if, it is compact, if, and only if,

$$\sum_{j\in\mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})} h(2^{-j})^{p'-1} < \infty, \quad \frac{1}{p}+\frac{1}{p'} = 1. \tag{1.3}$$

The above isotropic measure μ with the generating function h is called *strongly isotropic* if there is a natural number k such that

$$h(2^{-j-k}) \leqslant \tfrac{1}{2} h(2^{-j}) \quad \text{for all } j\in\mathbb{N}_0.$$

Let H be the inverse function of h. If μ is strongly isotropic and if (1.3) is strengthened by

$$\sum_{j\geqslant J} 2^{-jp'(s-\frac{n}{p})} h(2^{-j})^{p'-1} \sim 2^{-Jp'(s-\frac{n}{p})} h(2^{-J})^{p'-1}, \quad J\in\mathbb{N}_0 \tag{1.4}$$

(where the equivalence constants are independent of J) then one obtains for the approximation numbers a_k of the compact operator tr_μ according to (1.2)

$$a_k \sim k^{-\frac{1}{p}} H(k^{-1})^{s-\frac{n}{p}}, \quad k\in\mathbb{N}$$

(Theorem 2).

It is the *second* aim of this paper to apply the above result to fractal elliptic operators of type

$$B_s = (-\Delta + id)^{-s}\circ\mu : H^s(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n), \tag{1.5}$$

where $-\Delta$ is the usual Laplacian in $\mathbb{R}^n$ and

$$H^s(\mathbb{R}^n) = B^s_2(\mathbb{R}^n), \quad 0<s\leqslant\frac{n}{2}$$

are the well-known Sobolev spaces. Let μ with (1.1) be strongly isotropic with respect to the generating function h and let (1.4) be specified by $p = p' = 2$, hence

$$\sum_{j \geqslant J} 2^{j(n-2s)} h(2^{-j}) \sim 2^{J(n-2s)} h(2^{-J}), \quad J \in \mathbb{N}_0.$$

Then B_s is a compact, non-negative, self-adjoint operator in $H^s(\mathbb{R}^n)$. Let ϱ_k be its positive eigenvalues ordered by decreasing magnitude. Then

$$\varrho_k \sim k^{-1} H(k^{-1})^{2s-n}, \quad k \in \mathbb{N} \tag{1.6}$$

(Theorem 3). Of peculiar interest is the limiting case $s = \frac{n}{2}$. Then one has for all strongly isotropic measures μ with (1.1) and the related operators $B_{\frac{n}{2}}$ according to (1.5) with $s = \frac{n}{2}$ the Weylian behaviour

$$\varrho_k \sim k^{-1}, \quad k \in \mathbb{N},$$

of the corresponding positive eigenvalues.

These roughly outlined main results of this paper contribute to several branches of recent research and, in particular, to their interrelations:

- *The study of compact* (*embedding*) *operators acting between function spaces of type* B^s_{pq} *and* F^s_{pq} *including diverse modifications and generalisations*. The degree of compactness is preferably expressed in terms of entropy numbers and approximation numbers. In [10] one finds a description of the situation in the middle of the 1990s including many references. As for the recent state of art concerning approximation numbers in the indicated spaces we refer to [5,6,9,15–17].
- *The study of measures*. As for the geometrical aspects we refer to [12,21,13]. The more recent analysis on fractal sets and measures may be found in [18]. In [33] we discussed the close connection between Radon measures, multifractal quantities and function spaces of the above type.
- *The study of the distribution of eigenvalues of elliptic differential operators*. This is one of the major theories in analysis since the beginning of the last century which started with Weyl [35,36]. The recent state of art of the spectral theory of regular and singular elliptic differential operators and pseudodifferential operators and of the respective techniques may be found in [25]. The step from regular and singular to fractal is characterised by the key word *fractal drum*. There are several aspects which we discussed in [29, Section 26]. Maybe the best known interpretation is a drum in the plane with fractal boundary resulting in the study of the (Dirichlet or Neumann) Laplacian in bounded domains in the plane with fractal boundary and related subjects according to [19,20]. On the other hand we dealt in [29, Chapter V], and [30, Chapter III], with fractal drums, more precisely *drums with fractal membranes*, resulting in operators of type (1.5) with $s = 1$ and $n = 2$, and their analysis, in particular the distribution of eigenvalues.

The present paper might be considered as a contribution to the emerging close relationship between the indicated topics. In [31] we surveyed this subject in a larger context, announcing there some results proved in the present paper in detail. A first step beyond [29,30] has been done in [34] and some results obtained there are now improved in a definitive way. But there is a significant difference between the respective parts in [29,30,34] on the one hand, and the present paper on the other hand. We always relied on quarkonial (or subatomic) decompositions in function spaces as developed in [29,30]. In [29,30,34] we used this technique to deal with entropy numbers of compact embeddings between function spaces, which results in estimates (from above) of the positive eigenvalues of operators of type B_s in (1.5). Now we rely on (closely related) wavelet frames in function spaces according to [32] (which in turn are based on [30]). This gives the possibility to replace entropy numbers of compact embeddings between function spaces by respective approximation numbers.

In other word, it is the *third* aim of this paper to present a new method to estimate approximation numbers of compact operators acting between function spaces based on wavelet frames. Although our approach to quarkonial decompositions in function spaces and related wavelet frames is different in detail from what is known in literature one might complement the above list of related subjects as follows.

- *The study of wavelets in function spaces*. Wavelets, wavelet bases, and wavelet frames, preferably investigated in spaces of type $L_2(\mathbb{R}^n)$ and $L_p(\mathbb{R}^n)$ with $1<p<\infty$, have also been considered in diverse other types of function spaces, including spaces of type B^s_{pq} and F^s_{pq}. We refer to the respective sections in [22,14,7,24,37]. Our own approach, quarkonial decompositions in function spaces, which started in [29,30] as an instrument to study entropy numbers, has been formalised and modified in [32] in the context of wavelet analysis, now suitable to handle also approximation numbers. As said, it is one of the main aims of this paper to present this new possibility.

The plan of the paper is the following. In Section 2, we collect definitions and some first assertions. Section 3 contains the main results. Proofs are shifted to Sections 4, starting in 4.1 with a description of the wavelet frames according to [32] as far as needed here. In Section 5, we add a few complements.

2. Some prerequisites

2.1. Basic notation

We use standard notation. Let $\mathbb{N}$ be the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{R}^n$ be euclidean n-space, where $n \in \mathbb{N}$. Put $\mathbb{R} = \mathbb{R}^1$, whereas $\mathbb{C}$ is the complex plane. As usual, $\mathbb{Z}$ is the collection of all integers. Furthermore, $\mathbb{Z}^n$, where $n \in \mathbb{N}$, denotes the lattice of all points $m = (m_1, \dots, m_n) \in \mathbb{R}^n$ with $m_j \in \mathbb{Z}$. The set $\mathbb{N}_0^n$ of all multi-indices consists of all points

$$\beta = (\beta_1, \dots, \beta_n) \text{ with } \beta_j \in \mathbb{N}_0 \text{ and } |\beta| = \sum_{j=1}^{n} \beta_j.$$

We use the equivalence $\sim$ in

$$a_k \sim b_k \quad \text{or} \quad \varphi(x) \sim \psi(x) \tag{2.1}$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 a_k \leqslant b_k \leqslant c_2 a_k \quad \text{or} \quad c_1 \varphi(x) \leqslant \psi(x) \leqslant c_2 \varphi(x)$$

for all admitted values of the discrete variable k or the continuous variable x, where $\{a_k\}$, $\{b_k\}$ are sequences of positive numbers and φ, ψ are positive functions. Given two Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and the natural embedding of X in Y is continuous.

Let $S(\mathbb{R}^n)$ be the Schwartz space of all complex-valued, rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. By $S'(\mathbb{R}^n)$ we denote its topological dual, the spaces of tempered distributions on $\mathbb{R}^n$.

Furthermore, $L_p(\mathbb{R}^n)$ with $1 \leqslant p < \infty$ is the standard complex Banach space with respect to the Lebesgue measure μ_L, normed by

$$\|f|L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \mu_L\,(dx) \right)^{\frac{1}{p}},$$

where we prefer $\mu_L(dx)$ in place of dx.

2.2. Some function spaces

If f is a locally integrable (complex-valued) function in $\mathbb{R}^n$ then

$$(\Delta_h^1 f)(x) = f(x+h) - f(x) \quad \text{where } x \in \mathbb{R}^n, \quad 0 \neq h \in \mathbb{R}^n,$$

and, iteratively, $\Delta_h^M = \Delta_h^1(\Delta_h^{M-1})$ if $M-1 \in \mathbb{N}$, are the usual differences. Recall that for any $s \in \mathbb{R}$,

$$I_s : f \mapsto (id - \Delta)^{\frac{s}{2}} f, \quad \text{where} \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$$

is the usual Laplace operator in $\mathbb{R}^n$, maps $S(\mathbb{R}^n)$ onto itself and $S'(\mathbb{R}^n)$ onto itself.

Definition 1. Let $1 < p < \infty$.

(i) Let $s > 0$ and let $N \in \mathbb{N}$ with $N > s$. Then $B_p^s(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$\|f|B_p^s(\mathbb{R}^n)\| = \|f|L_p(\mathbb{R}^n)\| + \left(\int_{|h| \leqslant 1} |h|^{-sp} \|\Delta_h^N f|L_p(\mathbb{R}^n)\|^p \frac{dh}{|h|^n} \right)^{\frac{1}{p}} \tag{2.2}$$

is finite.

(ii) Let $s \in \mathbb{R}$. Then

$$H_p^s(\mathbb{R}^n) = I_{-s} L_p(\mathbb{R}^n) \tag{2.3}$$

and

$$H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n). \tag{2.4}$$

Remark 1. Recall that

$$B^s_p(\mathbb{R}^n) = B^s_{pq}(\mathbb{R}^n) \quad \text{with } q = p$$

are special classical Besov spaces, normed by (2.2), whereas $H^s_p(\mathbb{R}^n)$ are well-known Sobolev spaces normed by

$$\|f|H^s_p(\mathbb{R}^n)\| = \|I_s f|L_p(\mathbb{R}^n)\|, \quad f \in H^s_p(\mathbb{R}^n)$$

with the distinguished Hilbert spaces $H^s(\mathbb{R}^n)$ and the classical Sobolev spaces

$$H^k_p(\mathbb{R}^n) = W^k_p(\mathbb{R}^n), \quad k \in \mathbb{N}, \quad 1 < p < \infty$$

as subclasses. Of course, all spaces are considered in the framework of $S'(\mathbb{R}^n)$. For different values of $N \in \mathbb{N}$ with $N > s$ in (2.2) one gets equivalent norms in $B^s_p(\mathbb{R}^n)$. But this is unimportant for our purpose and not indicated on the left-hand side of (2.2) (one might think of the smallest admitted N). We do not distinguish between equivalent norms in a given space. In the main bulk of this paper, we restrict ourselves to the above spaces. The only exception is the final complementary section 5 where the more general spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$ will be mentioned. The theory of these function spaces has been developed systematically in [26–28]. In particular, the specific formulation in part (i) of the above definition is covered by [28, Theorem 2.6.1, p. 140, Corollary 1, p. 142]. Recall that

$$H^s(\mathbb{R}^n) = H^s_2(\mathbb{R}^n) = B^s_2(\mathbb{R}^n), \quad s > 0. \tag{2.5}$$

2.3. Measures

We always assume that μ is a positive Radon measure in $\mathbb{R}^n$ with

$$\Gamma = \operatorname{supp} \mu \quad \text{compact}, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0, \tag{2.6}$$

where $|\Gamma|$ is the Lebesgue measure of Γ. Let $1 \leqslant p < \infty$. Then $L_p(\Gamma, \mu)$ is the usual complex Banach space, normed by

$$\|f|L_p(\Gamma,\mu)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \mu\,(dx)\right)^{\frac{1}{p}} = \left(\int_\Gamma |f(\gamma)|^p \mu\,(d\gamma)\right)^{\frac{1}{p}},$$

where we use likewise both notation. Since μ is assumed to be Radon, $S(\mathbb{R}^n)$ or, likewise, its restriction $S(\mathbb{R}^n)|\Gamma$ to Γ is dense in $L_p(\Gamma, \mu)$. If $f \in L_p(\Gamma, \mu)$ then f, or better the complex Radon measure $f\mu$, can be interpreted in the usual way as a tempered distribution $id_\mu f$,

$$(id_\mu f)(\varphi) = \int_\Gamma f(\gamma)\varphi(\gamma)\mu\,(d\gamma), \quad \varphi \in S(\mathbb{R}^n). \tag{2.7}$$

The linear *identification operator* id_μ maps $L_p(\Gamma,\mu)$ continuously in $S'(\mathbb{R}^n)$. We refer for details to [30, 9.2, pp. 122–124]. A ball in $\mathbb{R}^n$ centred at $x\in\mathbb{R}^n$ and of radius r is denoted by $B(x,r)$.

Definition 2. Let μ be a Radon measure in $\mathbb{R}^n$ according to (2.6).

(i) Then μ is called isotropic if there is a continuous strictly increasing function h on the interval $[0,1]$ with $h(0)=0,\ h(1)=1$, and

$$\mu(B(\gamma,r))\sim h(r), \quad \gamma\in\Gamma=\operatorname{supp}\mu, \quad 0<r<1. \tag{2.8}$$

Then Γ is called a h-set.

(ii) The isotropic measure μ according to part (i) is called strongly isotropic if there is a natural number k such that

$$h(2^{-j-k})\leqslant\tfrac{1}{2}h(2^{-j}) \quad \text{for all } j\in\mathbb{N}_0. \tag{2.9}$$

Then Γ is called a strong h-set.

Remark 2. As for the use of $\sim$ we refer to (2.1). By $h(0)=0$ we exclude measures with atoms. Hence μ is *diffuse* according to [1, Section 5.10, p. 61]. In [30, p. 277], we called a Radon measure μ in the plane with (2.6), satisfying the doubling condition, *strongly diffuse* if there is a number λ with $0<\lambda<1$, such that

$$\mu(Q_1)\leqslant\tfrac{1}{2}\mu(Q_0)$$

for any cube Q_0 centred at some point $\gamma_0\in\Gamma=\operatorname{supp}\mu$ and of side-length r with $0<r<1$, and any sub-cube Q_1 with $Q_1\subset Q_0$ centred at some point $\gamma_1\in\Gamma$ and of side-length λr. The extension of this definition from $\mathbb{R}^2$ to $\mathbb{R}^n$ is obvious. It is quite clear that any strongly isotropic measure according to the above definition is in particular strongly diffuse what may justify this notation. The assumption $h(1)=1$ is convenient but immaterial. The almost classical example nowadays of strong h-sets are d-sets with $h(r)=r^d$ where $0<d<n$, hence

$$\mu(B(\gamma,r))\sim r^d, \quad \gamma\in\Gamma, \quad 0<r<1. \tag{2.10}$$

Details and references may be found in [29, pp. 5–7]. Perturbed d-sets, so-called (d,Ψ)-sets where $h(r)=r^d\Psi(r)$, typically with $\Psi(r)=|\log\frac{r}{2}|^b$ for some $b\in\mathbb{R}$, have been introduced in [11] and considered in detail in [23,17]. Arbitrary h-sets have been studied in [2–4]. It comes out that for a given function h with the above properties there is a compact set Γ and a Radon measure μ with (2.8) if, and only if, there exists an equivalent function $h^*,\ h\sim h^*$, with

$$h^*(2^{-j})\leqslant 2^{kn}h^*(2^{-j-k}) \quad \text{for all } j\in\mathbb{N}_0 \text{ and all } k\in\mathbb{N}_0. \tag{2.11}$$

One has in addition

$$|\Gamma|=0, \quad \text{if, and only if,} \quad \lim_{r\to 0} r^n h^{-1}(r)=0.$$

Proposition 1. *Let μ be an isotropic measure according to Definition* 2(i) *with the generating function h. Then the following three assertions are equivalent to each other*:

1. *μ is strongly isotropic*,

2. $$\sum_{j \geqslant J} h(2^{-j}) \sim h(2^{-J}) \quad \textit{for all } J \in \mathbb{N}_0, \tag{2.12}$$

3. $$\sum_{j \leqslant J} h^{-1}(2^{-j}) \sim h^{-1}(2^{-J}) \quad \textit{for all } J \in \mathbb{N}_0. \tag{2.13}$$

Proof. *Step* 1: Assume that μ is strongly isotropic according to (2.9). Then we have for $l \in \mathbb{N}_0$,

$$h(2^{-J-lk}) \leqslant 2^{-l} h(2^{-J}), \quad J \in \mathbb{N}_0$$

and with $J - lk \in \mathbb{N}_0$,

$$h^{-1}(2^{-J+lk}) \leqslant 2^{-l} h^{-1}(2^{-J}), \quad J \in \mathbb{N}_0.$$

Together with (2.11) and $h \sim h^*$ one gets (2.12) and (2.13).

Step 2: Assume that we have (2.12) and for some $J \in \mathbb{N}_0$ and $L \in \mathbb{N}$,

$$h(2^{-J-l}) \geqslant \tfrac{1}{2} h(2^{-J}) \quad \text{for } l = 0, \ldots, L.$$

Then

$$\frac{L+1}{2} h(2^{-J}) \leqslant \sum_{m=0}^{\infty} h(2^{-J-m}) \leqslant c h(2^{-J})$$

and $L + 1 \leqslant 2c$. Since h is monotone it follows that

$$h(2^{-J-L-1}) \leqslant \tfrac{1}{2} h(2^{-J}) \quad \text{for all } J \in \mathbb{N}_0$$

and hence (2.9).

Step 3: Assume that we have (2.13) and that for some $L \in \mathbb{N}$ and $J \geqslant L$,

$$h^{-1}(2^{-J+l}) \geqslant \tfrac{1}{2} h^{-1}(2^{-J}) \quad \text{for } l = 0, \ldots, L.$$

Then

$$\frac{L+1}{2} h^{-1}(2^{-J}) \leqslant \sum_{m=0}^{J} h^{-1}(2^{-m}) \leqslant c h^{-1}(2^{-J}).$$

One obtains (2.9) for some $k \in \mathbb{N}$ by the same arguments as above. $\square$

2.4. Traces

Again let μ be a Radon measure in $\mathbb{R}^n$ according to (2.6) and let $B^s_p(\mathbb{R}^n)$ be the spaces introduced in Definition 1. Let $1 \leqslant r < \infty$. We ask whether there is a constant $c>0$ such that

$$\left(\int_\Gamma |\varphi(\gamma)|^r \mu(d\gamma)\right)^{\frac{1}{r}} \leqslant c\|\varphi|B^s_p(\mathbb{R}^n)\| \quad \text{for all } \varphi \in S(\mathbb{R}^n). \tag{2.14}$$

If this is the case then one can extend (2.14) from $S(\mathbb{R}^n)$ to $B^s_p(\mathbb{R}^n)$ by completion using that $S(\mathbb{R}^n)$ is dense in $B^s_p(\mathbb{R}^n)$, [27, Theorem 2.3.3, p. 48]. Then any $f \in B^s_p(\mathbb{R}^n)$ has a (uniquely determined) trace $tr_\mu f \in L_r(\Gamma,\mu)$ and

$$tr_\mu : B^s_p(\mathbb{R}^n) \hookrightarrow L_r(\Gamma,\mu)$$

is denoted as the (linear and continuous) trace operator. We studied in [30, Section 9], in detail traces of function spaces of the above type. We collect a few easy consequences which will be of some service later on. Let Q_{jm} be the cubes in $\mathbb{R}^n$ with sides parallel to the axes of coordinates, centred at $2^{-j}m$ and with side length 2^{-j+1} where $m \in \mathbb{Z}^n$ and $j \in \mathbb{N}_0$. Let

$$\mu_j = \sup_{m \in \mathbb{Z}^n} \mu(Q_{jm}), \quad j \in \mathbb{N}_0,$$

what in case of isotropic measures simply means $\mu_j \sim h(2^{-j})$.

Proposition 2. *Let μ be a Radon measure in $\mathbb{R}^n$ according to* (2.6) *and let*

$$1<p<\infty, \quad \frac{1}{p}+\frac{1}{p'}=1, \quad s>0.$$

If

$$tr_\mu : B^s_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma,\mu) \tag{2.15}$$

exists (*as a linear and continuous map*) *then* (*necessary condition*)

$$\sum_{j\in\mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})} \sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^{p'} < \infty. \tag{2.16}$$

Conversely if

$$\sum_{j\in\mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})} \mu_j^{p'-1} < \infty \tag{2.17}$$

then tr_μ according to (2.15) *exists* (*sufficient condition*).

Proof. If tr_μ exists according to (2.15) then tr_μ is also a bounded map from $B^s_p(\mathbb{R}^n)$ into $L_1(\Gamma,\mu)$. Then (2.16) follows from [30, Theorem 9.9(ii), p. 131]. Conversely, since $p'-1=\frac{p'}{p}$, condition (2.17) coincides with [30, (9.47), p. 130], with $r=p$. This proves (2.15). $\square$

Remark 3. By

$$\sum_{m\in\mathbb{Z}^n} \mu(Q_{jm})^{p'} \leqslant \mu_j^{p'-1} \sum_{m\in\mathbb{Z}^n} \mu(Q_{jm}) \sim \mu_j^{p'-1}$$

it follows that the two conditions (2.16) and (2.17) are near to each other. They coincide if μ is isotropic. Of interest are only spaces $B^s_p(\mathbb{R}^n)$ with $s \leqslant \frac{n}{p}$ because otherwise the above conditions are automatically satisfied (recall that $B^s_p(\mathbb{R}^n)$ with $s > \frac{n}{p}$ is continuously embedded in $C(\mathbb{R}^n)$).

Corollary 1. *Let μ be an isotropic Radon measure according to Definition* 2(i) *with the generating function h. Let*

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 1 \leqslant r \leqslant p \ \text{ and } \ s > 0.$$

Then

$$tr_\mu : B^s_p(\mathbb{R}^n) \hookrightarrow L_r(\Gamma, \mu)$$

exists if, and only if,

$$\sum_{j \in \mathbb{N}_0} 2^{-jp'(s - \frac{n}{p})} h(2^{-j})^{p'-1} < \infty. \tag{2.18}$$

Proof. This assertion follows in the cases $r = 1$ and $r = p$ from [30, Theorem 9.9(ii), p. 131], and the above Remark 3, respectively. The rest is a matter of the monotonicity of the spaces $L_r(\Gamma, \mu)$. $\square$

2.5. The operators B_s and Weyl measures

We describe what is meant by the operator B_s mentioned in (1.5). Let $s > 0$ and let $H^s(\mathbb{R}^n)$ be the Sobolev spaces introduced in Definition 1(ii) and let μ be an isotropic Radon measure with the generating function h according to Definition 2(i) such that

$$\sum_{j \in \mathbb{N}_0} 2^{-j(2s-n)} h(2^{-j}) < \infty. \tag{2.19}$$

Then we have by Corollary 1 and (2.5) that

$$tr_\mu : H^s(\mathbb{R}^n) \hookrightarrow L_2(\Gamma, \mu) \tag{2.20}$$

(linear and continuous operator). Let id_μ be the identification operator according to (2.7). By [30, 9.2, pp. 122–125], the operators tr_μ and id_μ are dual to each other,

$$tr'_\mu = id_\mu \quad \text{and} \quad id'_\mu = tr_\mu, \tag{2.21}$$

hence, identifying as usual $L_2(\Gamma, \mu)$ with its dual, one gets by (2.20) within the dual pairing $(S(\mathbb{R}^n), S'(\mathbb{R}^n))$, that

$$id_\mu : L_2(\Gamma, \mu) \hookrightarrow H^{-s}(\mathbb{R}^n)$$

and, as a consequence,

$$id^\mu = id_\mu \circ tr_\mu : H^s(\mathbb{R}^n) \hookrightarrow H^{-s}(\mathbb{R}^n). \tag{2.22}$$

The precise understanding of (1.5) is now given by

$$B_s = (-\Delta + id)^{-s} \circ id^{\mu}. \tag{2.23}$$

Using (2.22) and well-known mapping properties it follows that

$$B_s : H^s(\mathbb{R}^n) \hookrightarrow H^s(\mathbb{R}^n) \tag{2.24}$$

is a linear and bounded operator in the Hilbert space $H^s(\mathbb{R}^n)$. We equip $H^s(\mathbb{R}^n)$ with the scalar product

$$(f, g)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (-\Delta + id)^{\frac{s}{2}} f(x) \cdot \overline{(-\Delta + id)^{\frac{s}{2}} g(x)} \mu_L(dx) \tag{2.25}$$

where again μ_L stands for the Lebesgue measure in $\mathbb{R}^n$, and where both f and g are elements of $H^s(\mathbb{R}^n)$. Extending the reasoning in [30, 19.3, p. 257], and in [29, 28.6, 30.2, pp. 226, 234], to the above case it follows that:

$$(B_s f, g)_{H^s(\mathbb{R}^n)} = \int_{\Gamma} (tr_\mu f)(\gamma) \cdot \overline{(tr_\mu g)(\gamma)} \mu(d\gamma).$$

Hence, B_s is a linear operator in $H^s(\mathbb{R}^n)$, generated by the scalar product in $L_2(\Gamma, \mu)$, considered as a bounded, non-negative, self-adjoint operator in $H^s(\mathbb{R}^n)$, and as a consequence,

$$\left\| \sqrt{B_s} f | H^s(\mathbb{R}^n) \right\| = \| tr_\mu f | L_2(\Gamma, \mu) \|, \quad f \in H^s(\mathbb{R}^n). \tag{2.26}$$

Remark 4. In the present paper, we restrict our attention to operators B_s where the underlying measure μ is isotropic. But this is not necessary. If μ is a Radon measure according to (2.6) with (2.20) then the above considerations remain valid. Criteria (necessary and sufficient conditions) for the existence of the trace operator tr_μ according to (2.20) in this general case may be found in [30, 9.3, 9.4, pp. 125–127]. Operators of type (2.23) have been considered several times, sometimes with the Dirichlet Laplacian $-\Delta$ in bounded smooth domains in $\mathbb{R}^n$ in place of $-\Delta + id$ in $\mathbb{R}^n$, but always under some restrictions for s and μ: Cases of preference are $s = 1$ (if in addition $n = 2$, then one arrives at drums with fractal membranes), d-sets according to (2.10) or (d, Ψ)-sets mentioned after (2.10). We refer to [29, Chapter V], [30, Chapter III], and [23]. In all cases considered, the above operator B_s is compact. Let $\{\varrho_k\}_{k \in \mathbb{N}}$ be the sequence of all positive eigenvalues of B_s, repeated according to multiplicity and ordered so that

$$\varrho_1 \geqslant \varrho_2 \geqslant \cdots > 0.$$

One of the main points in all these considerations is the determination of equivalences of type (1.6) for these eigenvalues. Of peculiar interest is the case

$$\varrho_k \sim k^{-1}, \quad k \in \mathbb{N}. \tag{2.27}$$

This is the classical Weyl behaviour which occurs if $s = 1,\ n = 2$ (fractal drums in the plane) and $s = \frac{n}{2}$ for general $n \in \mathbb{N}$. The latter case has been considered in [34]. But the main point in the present paper (as far as the operator B_s is concerned) is not so much that we now deal with general operators of type B_s but that our treatment is now based on approximation numbers, in sharp contrast to the above literature which relies mainly on entropy numbers.

Definition 3. An isotropic Radon measure μ according to (2.6) with the generating function h and

$$\sum_{j \in \mathbb{N}_0} h(2^{-j}) < \infty \tag{2.28}$$

is called a Weyl measure if

$$B_{\frac{n}{2}} = (-\Delta + id)^{-\frac{n}{2}} \circ id^{\mu}$$

according to (2.23), (2.24) is compact and if one has (2.27) for the positive eigenvalues ϱ_k of $B_{\frac{n}{2}}$.

Remark 5. Obviously, (2.28) is a special case of (2.18) or, better, (2.19). It comes out that all the above operators B_s, in particular $B_{\frac{n}{2}}$ in the above definition, are compact. In other words, the main (and only) point of the above definition is the distinguished distribution (2.27) of the positive eigenvalues.

3. Main results

3.1. Traces

All notation have the above meaning. In particular, the spaces $B^s_p(\mathbb{R}^n)$ have been introduced in Definition 1 and we explained at the beginning of Section 2.4 what is meant by the trace operator tr_μ and by the cubes Q_{jm} in $\mathbb{R}^n$.

Proposition 3. *Let*

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad s > 0.$$

Let μ be a Radon measure in $\mathbb{R}^n$ with

$$\Gamma = \operatorname{supp} \mu \quad \textit{compact}, \quad 0 < \mu(\mathbb{R}^n) < \infty, \quad |\Gamma| = 0, \tag{3.1}$$

and

$$\sum_{j \in \mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})} \mu_j^{p'-1} < \infty \quad \textit{where} \quad \mu_j = \sup_{m \in \mathbb{Z}^n} \mu(Q_{jm}). \tag{3.2}$$

Then tr_μ,

$$tr_\mu : B^s_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma, \mu) \tag{3.3}$$

is compact. Furthermore there is a constant c (depending on p and s) such that for all measures μ with (3.1), (3.2),

$$||tr_\mu|| \leqslant c \left(\sum_{j \in \mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})} \mu_j^{p'-1} \right)^{\frac{1}{p'}}. \tag{3.4}$$

Remark 6. Compared with Proposition 2 we have now estimate (3.4) and, more important, the assertion that tr_μ is compact. The proof is shifted to Section 4.2. It is based on wavelet frames according to [30,32]. It is just one of the main aims of this paper to use this new method to obtain assertions of the above type and estimates for the approximation numbers of the compact operator tr_μ. Accepting Proposition 3, the following assertion is an immediate consequence of Corollary 1.

Theorem 1. *Let*

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad s > 0$$

and let μ be an isotropic Radon measure μ according to Definition 2(i) *with the generating function h. Then the following three assertions are equivalent to each other*:

1. *The trace operator tr_μ,*

$$tr_\mu : B_p^s(\mathbb{R}^n) \hookrightarrow L_p(\Gamma, \mu) \tag{3.5}$$

exists,

2. *tr_μ is compact*,

3. $$\sum_{j \in \mathbb{N}_0} 2^{-jp'(s-\frac{n}{p})} h(2^{-j})^{p'-1} < \infty. \tag{3.6}$$

Proof. By Corollary 1, assertions 1 and 3 are equivalent. The compactness follows from Proposition 3. $\square$

Remark 7. If $s > \frac{n}{p}$ then (3.2) and (3.6) are always satisfied and Proposition 3 and Theorem 1 do not say very much. Hence the case of interest is $s \leqslant \frac{n}{p}$.

3.2. Approximation numbers

Let A and B be two Banach spaces and let $L(A, B)$ be the canonically normed Banach space of all linear and bounded operators acting from A to B. Let $T \in L(A, B)$. Then given any $k \in \mathbb{N}$, the kth approximation number $a_k(T)$ of T is defined by

$$a_k(T) = \inf\{||T - L||: \; L \in L(A, B), \; \text{rank}\, L < k\},$$

where rank L is the dimension of the range of L. This is a well-known notation and might be found in many books. We refer, for example, to [10, 1.3.1], and [8, II.2]. In particular, the degree of compactness of T can be measured how rapid $a_k(T)$ tends to zero. Let $T = tr_\mu$ according to Proposition 3. We strengthen (3.2) by

$$\sum_{j \geqslant J} 2^{-jp'(s-\frac{n}{p})} \mu_j^{p'-1} \sim 2^{-Jp'(s-\frac{n}{p})} \mu_J^{p'-1}, \quad J \in \mathbb{N}_0. \tag{3.7}$$

(According to the agreement in 2.1 the equivalence constants are independent of $J \in \mathbb{N}_0$). Again $s \leqslant \frac{n}{p}$ are the cases of interest, otherwise (3.7) is always satisfied. To avoid awkward notation let $a_t = a_{[t]}$ if $t \geqslant 1$ for the approximation numbers.

Proposition 4. *Let*

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad s > 0.$$

Let μ be a Radon measure in $\mathbb{R}^n$ with (3.1) *and* (3.7). *Let $a_k = a_k(tr_\mu)$ be the approximation numbers of the compact operator tr_μ in* (3.3). *There are two positive numbers c and c' such that*

$$a_{c2^{nJ}} \leqslant c' 2^{-J(s-\frac{n}{p})} \mu_J^{\frac{1}{p}}, \quad J \in \mathbb{N}_0. \tag{3.8}$$

Remark 8. Since $p' - 1 = \frac{p'}{p}$ it follows by (3.2) or (3.7) that the numbers on the right-hand side of (3.8) are tending to zero if J tends to infinity. We shift the proof of this proposition to 4.3. Otherwise (3.8) is a rather crude estimate. It μ is isotropic according to Definition 2 with the generating function h then one gets much better assertions. Let H be the inverse function of h, hence

$$h(t) = \tau \Leftrightarrow t = H(\tau), \quad 0 \leqslant t \leqslant 1, \quad 0 \leqslant \tau \leqslant 1. \tag{3.9}$$

Theorem 2. *Let*

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{p'} = 1, \quad 0 < s \leqslant \frac{n}{p}.$$

Let μ be a strongly isotropic Radon measure μ according to Definition 2(ii) *with the generating function h and the inverse function H, satisfying*

$$\sum_{j \geqslant J} 2^{-jp'(s-\frac{n}{p})} h(2^{-j})^{p'-1} \sim 2^{-Jp'(s-\frac{n}{p})} h(2^{-J})^{p'-1}, \quad J \in \mathbb{N}. \tag{3.10}$$

Let $a_k = a_k(tr_\mu)$ be the approximation numbers of the compact operator tr_μ according to (3.5). *Then*

$$a_k \sim k^{-\frac{1}{p}} H(k^{-1})^{s-\frac{n}{p}}, \qquad k \in \mathbb{N}. \tag{3.11}$$

Remark 9 (Example). This is a rather satisfactory assertion. The proof is shifted to 4.4. Now one can check diverse admitted functions h, maybe taken from the lists in [3,4]. This will not be done here. We restrict ourselves to the almost classical example of a compact d-set Γ with $0<d<n$,

$$h(t) = t^d \text{ and } H(t) = t^{\frac{1}{d}}, \quad 0 \leqslant t \leqslant 1.$$

Then we have (2.6) and $\mu = \mathcal{H}^d | \Gamma$ (the restriction of the Hausdorff measure $\mathcal{H}^d$ to Γ) is strongly isotropic. If

$$1<p<\infty, \quad \frac{1}{p}+\frac{1}{p'}=1, \quad \frac{n-d}{p}<s\leqslant\frac{n}{p},$$

then (3.10) is satisfied (recall that $p'-1=\frac{p'}{p}$) and we obtain

$$a_k \sim k^{-\frac{1}{p}} k^{-\frac{1}{d}(s-\frac{n}{p})} = k^{-\frac{1}{d}(s-\frac{n-d}{p})}, \quad k \in \mathbb{N}.$$

We remark that one has the same behaviour for the entropy numbers e_k of the trace operator tr_μ in (3.5). This follows from [29, 20.6, 20.2, pp. 166/159].

3.3. Fractal elliptic operators

We wish to apply the Theorems 1 and 2 with $p=2$ to the operators B_s introduced in Section 2.5. Recall that $H^s(\mathbb{R}^n) = B^s_2(\mathbb{R}^n)$ with $s>0$ are the Sobolev spaces according to (2.4) in Definition 1(ii) and (2.5). Let μ be an isotropic Radon measure according to Definition 2(i) with the generating function h. By Theorem 1, the trace operator tr_μ,

$$tr_\mu : H^s(\mathbb{R}^n) \hookrightarrow L_2(\Gamma, \mu)$$

exists, and is compact, if, and only, if,

$$\sum_{j\in\mathbb{N}_0} 2^{j(n-2s)} h(2^{-j}) < \infty.$$

In particular, the constructions described in Section 2.5 concerning the operator B_s can be applied and one gets the following assertion.

Theorem 3. *Let μ be a strongly isotropic Radon measure according to Definition* 2(ii) *with the generating function h and its inverse function H given by* (3.9). *Let $0<s\leqslant\frac{n}{2}$ and*

$$\sum_{j\geqslant J} 2^{j(n-2s)} h(2^{-j}) \sim 2^{J(n-2s)} h(2^{-J}), \quad J \in \mathbb{N}_0. \tag{3.12}$$

Then

$$B_s = (id - \Delta)^{-s} \circ id^\mu \tag{3.13}$$

according to Section 2.5 *is a compact, non-negative self-adjoint operator in $H^s(\mathbb{R}^n)$. Let $\{\varrho_k\}_{k\in\mathbb{N}}$ be the sequence of all positive eigenvalues of B_s, repeated according* *to multiplicity and ordered so that*

$$\varrho_1 \geqslant \varrho_2 \geqslant \cdots > 0, \quad \varrho_k \to 0 \quad \textit{if} \ k \to \infty. \tag{3.14}$$

Then

$$\varrho_k \sim k^{-1} H(k^{-1})^{2s-n}, \quad k \in \mathbb{N}. \tag{3.15}$$

Remark 10 (Example). The proof is shifted to Section 4.5. After the explanations given in Section 2.5 it remains to prove the distribution (3.15) of the eigenvalues ϱ_k. In case of a compact d-set Γ with $0<d<n$ we obtain by the same arguments as in Remark 9,

$$\varrho_k \sim k^{-\frac{2s-n+d}{d}}, \quad k \in \mathbb{N}$$

for the eigenvalues ϱ_k of B_s given by (3.13) with $n-d<2s\leqslant n$. We obtained this distribution of the eigenvalues in [29, Theorem 28.6, p. 226], in a slightly different but nearby context.

We introduced in Section 2.5, Definition 3, Weyl measures.

Corollary 2. *Any strongly isotropic Radon measure μ in $\mathbb{R}^n$ according to Definition* 2(ii) *is a Weyl measure.*

Proof. We apply Theorem 3 with $s=\frac{n}{2}$. Since μ is strongly isotropic it follows by (2.12) in Proposition 1 that (3.12) with $n=2s$ is satisfied. Hence $\varrho_k \sim k^{-1}$ where $k \in \mathbb{N}$ and μ is a Weyl measure. □

Remark 11. As mentioned in Remark 2, any strongly isotropic measure is a strongly diffuse Radon measure. We proved in [30, Theorem 19.17, p. 280], that any finite, strongly diffuse, compactly supported Radon measure in the plane $\mathbb{R}^2$ is a Weyl measure. The somewhat complicated proof is mainly based on entropy numbers. But there is hardly any doubt (although not done in detail so far) that this proof can be extended to $n \in \mathbb{N}$. Then the above corollary would be a special case of such an assertion. But here it is a simple by-product of Theorem 3. The first step to deal with Weyl measures in $\mathbb{R}^n$ had been done in [34] based on entropy numbers. Using the above Proposition 1 it comes out that Theorem 1 in [34] is a special case of the above Corollary 2. But both techniques, based on entropy numbers or approximation numbers, respectively, have their advantages and disadvantages (and we were not aware of the above simple Proposition 1 when [34] was written).

4. Proofs

It remains to prove Propositions 3 and 4, Theorems 2 and 3. But first we describe wavelet frames which are our basic tool in what follows. As mentioned in the

Introduction it is one of the main aims of this paper to present this method as an effective instrument to estimate approximation numbers.

4.1. Wavelet frames

We developed the theory of quarkonial (or subatomic) decompositions in function spaces of type B^s_{pq} and F^s_{pq} with $s \in \mathbb{R}$, $0 < p \leqslant \infty$ ($p < \infty$ in the F-case), $0 < q \leqslant \infty$, in [29, Section 14], and in [30, Sections 2 and 3], mainly as an instrument to estimate entropy numbers of compact operators between these spaces in $\mathbb{R}^n$, in domains, on manifolds and on fractals. We returned to this subject in [32] and used it in connection with a global, local and pointwise regularity theory for distributions. We follow here this paper in a slightly modified way restricting ourselves to the special case

$$B^s_p(\mathbb{R}^n) \quad \text{with } 1 < p < \infty, \quad s > 0,$$

according to Definition 1(i).

Let $\varkappa$ be a non-negative C^∞ function in $\mathbb{R}^n$ with

$$\operatorname{supp} \varkappa \subset \{y = (y_1, \ldots, y_n) \in \mathbb{R}^n\colon\ |y| < 2^N,\ y_j > 0\} \tag{4.1}$$

for some fixed $N \in \mathbb{N}$ and

$$\sum_{m \in \mathbb{Z}^n} \varkappa(x - m) = 1 \quad \text{where } x \in \mathbb{R}^n.$$

Let

$$\varkappa^\beta(x) = (2^{-N}x)^\beta \varkappa(x) \geqslant 0 \quad \text{if } x \in \mathbb{R}^n \quad \text{and} \quad \beta \in \mathbb{N}^n_0,$$

where $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$. Let

$$\omega \in S(\mathbb{R}^n), \quad \operatorname{supp} \omega \subset (-\pi, \pi)^n, \quad \omega(x) = 1 \text{ if } |x| \leqslant 2,$$

$$\omega^\beta(x) = \frac{i^{|\beta|} 2^{N|\beta|}}{(2\pi)^n \beta!} x^\beta \omega(x), \quad \text{where} \quad x \in \mathbb{R}^n \quad \text{and} \quad \beta \in \mathbb{N}^n_0$$

and

$$\Omega^\beta(x) = \sum_{m \in \mathbb{Z}^n} (\omega^\beta)^\vee(m) e^{-imx}, \quad x \in \mathbb{R}^n,$$

where $|\beta| = \beta_1 + \cdots + \beta_n$ and $= \beta_1! \cdots \beta_n!$. As usual, $^\wedge$ and $^\vee$ refer to the Fourier transform and its inverse, respectively. Let $\varphi_0 \in S(\mathbb{R}^n)$,

$$\varphi_0(x) = 1 \quad \text{if } |x| \leqslant 1, \quad \varphi_0(x) = 0 \quad \text{if } |x| \geqslant \tfrac{3}{2},$$

and $\varphi(x) = \varphi_0(x) - \varphi_0(2x)$. Then

$$\Phi^\beta_{jm}(x) = \begin{cases} \Phi^\beta_F(x - m) & \text{if } j = 0,\ m \in \mathbb{Z}^n, \\ \Phi^\beta_M(2^j x - m) & \text{if } j \in \mathbb{N},\ m \in \mathbb{Z}^n \end{cases}$$

are analytic wavelets where the father wavelets Φ_F^β and the mother wavelets Φ_M^β are given by their inverse Fourier transforms

$$(\Phi_F^\beta)^\vee(\xi) = \varphi_0(\xi)\Omega^\beta(\xi), \quad (\Phi_M^\beta)^\vee(\xi) = \varphi(\xi)\Omega^\beta(\xi),$$

with $\xi \in \mathbb{R}^n$. For the sequence

$$\lambda = \{\lambda_{jm}^\beta \in \mathbb{C} : j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ \beta \in \mathbb{N}_0^n\},$$

$s > 0,\ 1 < p < \infty$ and $\varrho \geqslant 0$ we put

$$\|\lambda|\ell_p\|_{\varrho,s} = \left(\sum_{\beta,j,m} 2^{\varrho|\beta|p + j(s-\frac{n}{p})p} |\lambda_{jm}^\beta|^p\right)^{\frac{1}{p}}, \tag{4.2}$$

where $\sum_{\beta,j,m}$ always means the summation over $\beta \in \mathbb{N}_0^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n$. For $f \in L_p(\mathbb{R}^n)$,

$$\lambda_{jm}^\beta(f) = 2^{jn} \int_{\mathbb{R}^n} f(x) \Phi_{jm}^\beta(x)\, dx, \quad j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n,\ \beta \in \mathbb{N}_0^n \tag{4.3}$$

are distinguished wavelet coefficients. We formulate now a special case of Theorem 1 in [32] in a slightly modified version. We put

$$\varkappa_{jm}^\beta(x) = \varkappa^\beta(2^j x - m), \quad \beta \in \mathbb{N}_0^n,\ j \in \mathbb{N}_0,\ m \in \mathbb{Z}^n.$$

Let $1 < p < \infty,\ s > 0$, and $\varrho \geqslant 0$. Then $f \in L_p(\mathbb{R}^n)$ is an element of $B_p^s(\mathbb{R}^n)$ if, and only if, it can be represented as

$$f = \sum_{\beta,j,m} \lambda_{jm}^\beta \varkappa_{jm}^\beta \tag{4.4}$$

with $\|\lambda|\ell_p\|_{\varrho,s} < \infty$. Furthermore,

$$\|f|B_p^s(\mathbb{R}^n)\| \sim \inf \|\lambda|\ell_p\|_{\varrho,s} \tag{4.5}$$

where the infimum is taken over all admissible representations (4.4). In addition, any $f \in B_p^s(\mathbb{R}^n)$ can be optimally represented by

$$f = \sum_{\beta,j,m} \lambda_{jm}^\beta(f) \varkappa_{jm}^\beta \tag{4.6}$$

with

$$\|f|B_p^s(\mathbb{R}^n)\| \sim \|\lambda(f)|\ell_p\|_{\varrho,s}. \tag{4.7}$$

The convergence in (4.4), (4.6) is absolute in $L_p(\mathbb{R}^n)$ and unconditional in $B_p^s(\mathbb{R}^n)$. Furthermore, $\varrho \geqslant 0$ can be prescribed where the equivalence constants in (4.5), (4.7) may depend on $s,\ \varrho,\ p$. We refer for further details to [29, Section 14; 30, Sections 2 and 3], and, in particular, to [32]. Of special interest for us is the wavelet representation (4.6) with (4.3), (4.7).

4.2. Proof of Proposition 3

Step 1: Let $f \in B^s_p(\mathbb{R}^n)$ be given by (4.6), (4.7). For any fixed $\beta \in \mathbb{N}^n_0$ we have

$$\left\| \sum_{j=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda^{\beta}_{jm}(f) \varkappa^{\beta}_{jm} | L_p(\Gamma, \mu) \right\| \leqslant c \sum_{j=0}^{\infty} \mu_j^{\frac{1}{p}} \left(\sum_{m \in \mathbb{Z}^n} |\lambda^{\beta}_{jm}(f)|^p \right)^{\frac{1}{p}}$$

$$\leqslant c \left(\sum_{j=0}^{\infty} 2^{-jp'(s-\frac{n}{p})} \mu_j^{\frac{p'}{p}} \right)^{\frac{1}{p'}} \left(\sum_{j,m} 2^{j(s-\frac{n}{p})p} |\lambda^{\beta}_{jm}(f)|^p \right)^{\frac{1}{p}},$$

where c is independent of β and μ. We choose $\varrho > 0$. Then it follows by (4.2), (4.7) and $\frac{p'}{p} = p' - 1$ that

$$\|tr_\mu f | L_p(\Gamma, \mu)\| \leqslant c \left(\sum_{j=0}^{\infty} 2^{-jp'(s-\frac{n}{p})} \mu_j^{p'-1} \right)^{\frac{1}{p'}} \|f | B^s_p(\mathbb{R}^n)\|,$$

where c is independent of μ. This proves (3.4).

Step 2: We prove that tr_μ is compact. Let $B \in \mathbb{N}$, $J \in \mathbb{N}$, and let $tr^{B,J}_\mu$ be given by

$$tr^{B,J}_\mu f = \sum_{|\beta| \leqslant B} \sum_{j \leqslant J} \sum_{m \in \mathbb{Z}^n}^{\Gamma} \lambda^{\beta}_{jm}(f) \varkappa^{\beta}_{jm}, \tag{4.8}$$

where again $f \in B^s_p(\mathbb{R}^n)$ is given by (4.6), (4.7) and where the sum $\sum^{\Gamma}_{m \in \mathbb{Z}^n}$ is restricted to those $m \in \mathbb{Z}^n$ such that the cubes Q_{jm} have a non-empty intersection with Γ. For given $b > 0$ and suitably chosen $\varrho > 0$ it follows by the above arguments for $f \in B^s_p(\mathbb{R}^n)$ having norm of at most 1 that

$$\begin{aligned} &\|(tr_\mu - tr^{B,J}_\mu) f | L_p(\Gamma, \mu)\| \\ &\quad \leqslant c \left(\sum_{|\beta| \geqslant B} 2^{-b|\beta|} \right) + c \left(\sum_{|\beta| \leqslant B} 2^{-b|\beta|} \right) \left(\sum_{j \geqslant J} 2^{-jp'(s-\frac{n}{p})} \mu_j^{p'-1} \right)^{\frac{1}{p'}}, \end{aligned} \tag{4.9}$$

where c is independent of f. By (3.2) we find for any $\varepsilon > 0$ given sufficiently large numbers B and J such that

$$\|tr_\mu - tr^{B,J}_\mu\| \leqslant \varepsilon.$$

Since $tr^{B,J}_\mu$ are operators of finite rank it follows that tr_μ is compact.

4.3. Proof of Proposition 4

By (3.7) we have (3.2). Hence by Proposition 3 the operator tr_μ is compact. We refine (4.8) by

$$tr^J_\mu f = \sum_{|\beta|\leqslant J} \sum_{j\leqslant J-|\beta|} \sum_{m\in\mathbb{Z}^n}^{\Gamma} \lambda^\beta_{jm}(f)\varkappa^\beta_{jm}, \quad J\in\mathbb{N}, \tag{4.10}$$

where again $f\in B^s_p(\mathbb{R}^n)$ is given by (4.6), (4.7) and the last sum has the same meaning as the last sum in (4.8). Since μ is a measure in $\mathbb{R}^n$ we have

$$\mu_K \leqslant c2^{(J-K)n}\mu_J \quad \text{where } K\leqslant J. \tag{4.11}$$

Let $b>0$ be sufficiently large. By (3.7) with $p'-1=\frac{p'}{p}$ and (4.11) we obtain for $f\in B^s_p(\mathbb{R}^n)$ having norm of at most 1 in analogy to (4.9) that

$$\begin{aligned}
&\|(tr_\mu - tr^J_\mu)f|L_p(\Gamma,\mu)\| \\
&\leqslant c2^{-bJ} + c\sum_{|\beta|\leqslant J} 2^{-b|\beta|}\left(\sum_{j\geqslant J-|\beta|} 2^{-jp'(s-\frac{n}{p})}\mu_j^{\frac{p'}{p}}\right)^{\frac{1}{p'}} \\
&\leqslant c2^{-bJ} + c\sum_{|\beta|\leqslant J} 2^{-b|\beta|}2^{-(J-|\beta|)(s-\frac{n}{p})}\mu^{\frac{1}{p}}_{J-|\beta|} \\
&\leqslant c2^{-bJ} + c\mu_J^{\frac{1}{p}}2^{-J(s-\frac{n}{p})}\sum_{|\beta|\leqslant J} 2^{-b|\beta|+\frac{n}{p}|\beta|+|\beta|(s-\frac{n}{p})} \\
&\leqslant c'2^{-J(s-\frac{n}{p})}\mu_J^{\frac{1}{p}}
\end{aligned} \tag{4.12}$$

for sufficiently large $b>0$. In the last estimate we used (4.11) with $K=0$. For the rank of tr^J_μ we have the somewhat crude estimate

$$\operatorname{rank}(tr^J_\mu) \leqslant c\sum_{|\beta|\leqslant J} 2^{n(J-|\beta|)} \leqslant c'2^{nJ}.$$

This proves (3.8).

4.4. Proof of Theorem 2

Step 1: First, we prove that there is a number $c>0$ such that

$$a_k(tr_\mu) \leqslant ck^{-\frac{1}{p}}H(k^{-1})^{s-\frac{n}{p}}, \quad k\in\mathbb{N}. \tag{4.13}$$

Again we rely on the wavelet expansion (4.6), (4.7). For fixed $\beta\in\mathbb{N}^n_0$ we put

$$tr^\beta_\mu f = \sum_{j\in\mathbb{N}_0} \sum_{m\in\mathbb{Z}^n} \lambda^\beta_{jm}(f)\varkappa^\beta_{jm} \tag{4.14}$$

and

$$tr_\mu^{\beta,J} f = \sum_{j \leqslant J} \sum_{m \in \mathbb{Z}^n}^{\Gamma} \lambda_{jm}^\beta(f) \varkappa_{jm}^\beta, \tag{4.15}$$

where the second sum in (4.15) has the same meaning as the last sum in (4.8). By the same reasoning as in (4.10), (4.12) but now for fixed β we have for $f \in B_p^s(\mathbb{R}^n)$ with norm of at most 1,

$$\|(tr_\mu^\beta - tr_\mu^{\beta,J})f | L_p(\Gamma,\mu)\| \leqslant c 2^{-b|\beta|} 2^{J(\frac{n}{p}-s)} h(2^{-J})^{\frac{1}{p}}, \tag{4.16}$$

where $b>0$ is at our disposal and $c>0$ is independent of $\beta \in \mathbb{N}_0^n$ and $J \in \mathbb{N}_0$ (but may depend on b). We used (3.10) as the specification of (3.7) with $\mu_j \sim h(2^{-j})$. Since μ is strongly isotropic it follows by Proposition 1 that

$$\operatorname{rank}(tr_\mu^{\beta,J}) \sim \sum_{j \leqslant J} h(2^{-j})^{-1} \sim h(2^{-J})^{-1}.$$

We obtain by (4.16) that there are two positive numbers c and c' such that for all $\beta \in \mathbb{N}_0^n$ and $J \in \mathbb{N}_0$,

$$a_{ch(2^{-J})^{-1}}(tr_\mu^\beta) \leqslant c' 2^{-b|\beta|} 2^{J(\frac{n}{p}-s)} h(2^{-J})^{\frac{1}{p}}. \tag{4.17}$$

Recall that $b>0$ is at our disposal and that c and c' may depend on b. By (2.11) and $h \sim h^*$ we have $h(2^{-j-1}) \sim h(2^{-j})$ where $j \in \mathbb{N}_0$. Hence for $k \in \mathbb{N}$ there are numbers $J_k \in \mathbb{N}$ such that

$$h(2^{-J_k})^{-1} \sim k \quad \text{with} \quad J_1 \leqslant J_2 \leqslant \cdots \leqslant J_k \leqslant \cdots \to \infty \tag{4.18}$$

if $k \to \infty$. Inserted in (4.17) one obtains

$$a_k(tr_\mu^\beta) \leqslant c 2^{-b|\beta|} 2^{J_k(\frac{n}{p}-s)} k^{-\frac{1}{p}}, \quad k \in \mathbb{N}. \tag{4.19}$$

Let $\varepsilon>0$. For given $k \in \mathbb{N}$ we apply (4.19) to $k_\beta \in \mathbb{N}$ with $k_\beta \sim 2^{-\varepsilon|\beta|}k$ (this means 1 if the latter number is between 0 and 1). Then it follows from the additivity property of approximation numbers and from (4.19) that

$$\begin{aligned} a_{ck}(tr_\mu) &\leqslant \sum_{\beta \in \mathbb{N}_0^n} a_{k_\beta}(tr_\mu^\beta) \\ &\leqslant c' \sum_{\beta \in \mathbb{N}_0^n} 2^{-b|\beta|} 2^{\frac{\varepsilon}{p}|\beta|} 2^{J_{k_\beta}(\frac{n}{p}-s)} k^{-\frac{1}{p}} \\ &\leqslant c'' 2^{J_k(\frac{n}{p}-s)} k^{-\frac{1}{p}}. \end{aligned} \tag{4.20}$$

We used $s \leqslant \frac{n}{p}$ and the monotonicity of the numbers J_k in (4.18). Now, (4.13) is a consequence of (4.20) and (4.18).

Step 2: We prove that for two suitable positive numbers c and c',

$$a_{ch(2^{-j})^{-1}}(tr_\mu) \geqslant c' 2^{-j(s-\frac{n}{p})} h(2^{-j})^{\frac{1}{p}}, \quad j \in \mathbb{N}_0. \tag{4.21}$$

By (4.18) this is equivalent to the converse of (4.13) and completes the proof of (3.11). We use the same type of arguments as in [29, pp. 219–220], appropriately modified. Let $J \in \mathbb{N}$ and $c > 0$ be suitably chosen numbers such that there are lattice points

$$\gamma^{j,l} \in 2^{-j-J}\mathbb{Z}^n, \quad l = 1, \dots, M_j \quad \text{where} \quad M_j \sim h(2^{-j})^{-1} \tag{4.22}$$

with

$$\mathrm{dist}(\gamma^{j,l}, \Gamma) \leqslant c2^{-j} \quad \text{and disjoint balls } B(\gamma^{j,l}, c2^{-j+1}). \tag{4.23}$$

Here $j \in \mathbb{N}_0$ and the (equivalence) constants in (4.22) and (4.23) are independent of j. With $\varkappa$ as in (4.1) we put for $j \in \mathbb{N}_0$,

$$f_j(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{p})} \varkappa(2^j(x - \gamma^{j,l})), \quad c_{jl} \in \mathbb{C}, \quad x \in \mathbb{R}^n. \tag{4.24}$$

Then we obtain by the localisation property according to [10, 2.3.2, pp. 35–36],

$$\|f_j | B^s_p(\mathbb{R}^n)\| \sim \left(\sum_{l=1}^{M_j} |c_{jl}|^p \right)^{\frac{1}{p}} \tag{4.25}$$

and (all constants are suitably chosen)

$$\|f_j | L_p(\Gamma, \mu)\| \sim 2^{-j(s-\frac{n}{p})} h(2^{-j})^{\frac{1}{p}} \left(\sum_{l=1}^{M_j} |c_{jl}|^p \right)^{\frac{1}{p}}. \tag{4.26}$$

All equivalence constants are independent of $j \in \mathbb{N}_0$. Hence,

$$\|f_j | L_p(\Gamma, \mu)\| \sim 2^{-j(s-\frac{n}{p})} h(2^{-j})^{\frac{1}{p}} \quad \text{if } \|f_j | B^s_p(\mathbb{R}^n)\| \sim 1. \tag{4.27}$$

Now let T be a linear operator,

$$T : B^s_p(\mathbb{R}^n) \hookrightarrow L_p(\Gamma, \mu) \quad \text{with} \quad \mathrm{rank}\, T \leqslant M_j - 1. \tag{4.28}$$

Then one finds a function f_j according to (4.24) with norm 1 in $B^s_p(\mathbb{R}^n)$ and $Tf_j = 0$. Hence, by (4.27),

$$a_{M_j}(tr_\mu) = \inf \|tr_\mu - T\| \geqslant c2^{-j(s-\frac{n}{p})} h(2^{-j})^{\frac{1}{p}}, \quad j \in \mathbb{N}_0, \tag{4.29}$$

where the infimum is taken over all T with (4.28). Here c is some positive constant which is independent of $j \in \mathbb{N}_0$. Now (4.21) follows from (4.29) and (4.22).

4.5. Proof of Theorem 3

Step 1: By the explanations given in Section 2.5,

$$B_s = (id - \Delta)^{-s} \circ id^\mu \quad \text{where} \quad id^\mu = id_\mu \circ tr_\mu, \tag{4.30}$$

is a bounded, non-negative self-adjoint operator in $H^s(\mathbb{R}^n)$ equipped with the scalar product (2.25). If $p = 2$ then (3.12) coincides with (3.10). Then it follows by Theorem

2 that B_s is compact. Let ϱ_k be the positive eigenvalues of B_s ordered according to (3.14). It remains to prove (3.15).

Step 2: We prove that there is a number $c>0$ such that

$$\varrho_k \leqslant c k^{-1} H(k^{-1})^{2s-n}, \quad k \in \mathbb{N}. \tag{4.31}$$

By (2.21) the identification operator id_μ is the dual of the trace operator tr_μ. By the usual assertion for dual operators, [8, Proposition II, 2.5, p. 55], and Theorem 2 we have

$$a_k(id_\mu) = a_k(tr_\mu) \sim k^{-\frac{1}{2}} H(k^{-1})^{s-\frac{n}{2}}, \quad k \in \mathbb{N}. \tag{4.32}$$

By (4.30) and the multiplication property for approximation numbers, [8, Proposition II, 2.2, p. 53], one obtains

$$a_{2k}(B_s) \sim a_{2k}(id^\mu) \leqslant a_k(tr_\mu) a_k(id_\mu) \sim k^{-1} H(k^{-1})^{2s-n}. \tag{4.33}$$

Recall that $a_k(B_s) = \varrho_k$, [8, Theorem II, 5.10, p. 91]. Then (4.31) follows from (4.33) and

$$H(2^{-j}) \sim H(2^{-j+1}), \quad j \in \mathbb{N}_0.$$

Step 3: Recall $a_k(\sqrt{B_s}) = \varrho_k^{\frac{1}{2}}$. Hence, to obtain the converse of (4.31) it is sufficient to prove that

$$a_k(\sqrt{B_s}) \geqslant c k^{-\frac{1}{2}} H(k^{-1})^{s-\frac{n}{2}}, \quad k \in \mathbb{N}, \tag{4.34}$$

for some $c>0$. We use the same arguments as in 4.4, Step 2, now with $p=2$. Hence, based on (4.22), (4.23), we put

$$f_j(x) = \sum_{l=1}^{M_j} c_{jl} 2^{-j(s-\frac{n}{2})} \varkappa(2^j(x - \gamma^{j,l})), \quad c_{jl} \in \mathbb{C}, \ x \in \mathbb{R}^n.$$

By (4.25)–(4.27), (2.5), and (2.26) we have

$$\left\| \sqrt{B_s} f_j | H^s(\mathbb{R}^n) \right\| \sim 2^{-j(s-\frac{n}{2})} h(2^{-j})^{\frac{1}{2}} \quad \text{if } \| f_j | H^s(\mathbb{R}^n) \| \sim 1.$$

By the same arguments as in connection with (4.28), (4.29) we obtain for $M_j \sim h(2^{-j})^{-1}$ that

$$a_{M_j}(\sqrt{B_s}) \geqslant c 2^{-j(s-\frac{n}{2})} h(2^{-j})^{\frac{1}{2}}, \quad j \in \mathbb{N}_0. \tag{4.35}$$

This is the counterpart of (4.21). Hence, as there, (4.34) is a consequence of (4.35).

5. Complements

We used wavelet frames as the main tool to estimate approximation numbers in some special function spaces. Although not done in detail so far this technique can be used also in other situations. But it is not our aim to comment on these possibilities. Just on the contrary. We add two corollaries which rely on the above assertions combined with some general results available in literature.

5.1. Interpolation

We assume that the reader of this subsection is familiar with the theory of the function spaces $B^s_{pq}(\mathbb{R}^n)$ and $F^s_{pq}(\mathbb{R}^n)$. Recall that

$$F^s_{p,2}(\mathbb{R}^n) = H^s_p(\mathbb{R}^n), \quad 1<p<\infty, \quad s\in\mathbb{R},$$

are the Sobolev spaces according to (2.3). As usual nowadays $A^s_{pq}(\mathbb{R}^n)$ stands either for $B^s_{pq}(\mathbb{R}^n)$ or $F^s_{pq}(\mathbb{R}^n)$ (indicating that the assertion in question is equally valid for B-spaces and F-spaces).

Corollary 3. *Let*

$$1<p<\infty, \quad \frac{1}{p}+\frac{1}{p'}=1, \quad 0<s_0<\frac{n}{p}, \; 0<q\leqslant\infty.$$

Let μ be a strongly isotropic Radon measure μ according to Definition 2(ii) *with the generating function h and its inverse function H, satisfying*

$$\sum_{j\geqslant J} 2^{-jp'(s_0-\frac{n}{p})} h(2^{-j})^{p'-1} \sim 2^{-Jp'(s_0-\frac{n}{p})} h(2^{-J})^{p'-1}, \quad J\in\mathbb{N}. \tag{5.1}$$

Let $s_0<s<\frac{n}{p}$. Then

$$tr_\mu : A^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma,\mu) \tag{5.2}$$

is compact and

$$a_k(tr_\mu) \sim k^{-\frac{1}{p}} H(k^{-1})^{s-\frac{n}{p}}, \quad k\in\mathbb{N}, \tag{5.3}$$

where $a_k(tr_\mu)$ are the respective approximation numbers.

Proof. Since

$$B^s_{p,\min(p,q)}(\mathbb{R}^n) \hookrightarrow F^s_{p,q}(\mathbb{R}^n) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n)$$

it is sufficient to deal with the B-spaces. Furthermore we have (5.1) with s_1 in place of s_0 where $s_0\leqslant s_1\leqslant\frac{n}{p}$. We wish to apply Theorem 2, or better its proof in 4.4. Recall the real interpolation formula

$$(B^{s_0}_p(\mathbb{R}^n), B^{s_1}_p(\mathbb{R}^n))_{\theta,q} = B^s_{pq}(\mathbb{R}^n)$$

where $0<q\leqslant\infty$,

$$s_0<s_1=\frac{n}{p}, \quad 0<\theta<1, \quad s=(1-\theta)s_0+\theta s_1$$

[26,27]. We apply the interpolation property to the universal operators tr^β_μ and $tr^{\beta,J}_\mu$ in (4.14) and (4.15) (which are independent of p and s). Then one obtains the

counterparts of (4.16), (4.17), and hence (4.13) now for

$$tr_\mu : B^s_{pq}(\mathbb{R}^n) \hookrightarrow L_p(\Gamma, \mu), \quad s_0 < s < \frac{n}{p}.$$

As for the converse one can use the same arguments as in Step 2 in 4.4. The main point is the application of the localisation property according to [10, 2.3.2, pp. 35–36]. But this works for all spaces $A^s_{pq}(\mathbb{R}^n)$. $\square$

5.2. Entropy numbers

We describe a second application of the above results: estimates of entropy numbers by means of approximation numbers. Let A and B be quasi-Banach spaces and let $T \in L(A, B)$; let U_A be the unit ball in A. Then for $k \in \mathbb{N}$, the kth entropy number $e_k(T)$ of T is defined as the infimum of all $\varepsilon > 0$ such that $T(U_A)$ can be covered by 2^{k-1} balls of radius ε in B. Otherwise we assume that the reader of this subsection is familiar with entropy numbers and their use.

Corollary 4. *Let μ be a strongly isotropic Radon measure μ according to Definition* 2(ii) *with the generating function h and its inverse function H.*

(i) *Let the hypotheses of Theorem* 2 *be satisfied and let $e_k(tr_\mu)$ be the entropy numbers of tr_μ according to* (3.5). *Then there is a positive number c such that*

$$e_k(tr_\mu) \leqslant c k^{-\frac{1}{p}} H(k^{-1})^{s-\frac{n}{p}}, \quad k \in \mathbb{N}. \tag{5.4}$$

(ii) *Let the hypotheses of Corollary* 3 *be satisfied. Then the entropy numbers $e_k(tr_\mu)$ of tr_μ now given by* (5.2) *can be estimated according to* (5.4).

(iii) *Let the hypotheses of Theorem* 3 *be satisfied. Let*

$$tr_\mu : H^s(\mathbb{R}^n) \hookrightarrow L_2(\Gamma, \mu).$$

Then

$$e_k(tr_\mu) \sim a_k(tr_\mu) \sim k^{-\frac{1}{2}} H(k^{-1})^{s-\frac{n}{2}}, \quad k \in \mathbb{N}. \tag{5.5}$$

Proof. *Step* 1: By (3.11) we have $a_{2^{j-1}} \sim a_{2^j}$ where $j \in \mathbb{N}_0$. Then it follows by [10, Theorem 1.3.3, p. 15], that

$$e_k(tr_\mu) \leqslant c a_k(tr_\mu), \quad k \in \mathbb{N}. \tag{5.6}$$

Hence (i), and similarly (ii), follow from (3.11) and (5.3), respectively.

Step 2: The last equivalence in (5.5) is covered by (3.11). We have also (5.6). As for the converse estimate we remark that

$$e_k(id_\mu) = e_k(tr_\mu), \quad k \in \mathbb{N}.$$

This follows in analogy to (4.32) from the duality assertion for entropy numbers in Hilbert spaces, [10, Theorem 1.3.1, p. 9]. Similarly as in (4.33) one obtains now by the multiplication property for entropy numbers and by Carl's inequality, [10, Corollary 1.3.4, p. 20],

$$\begin{aligned} k^{-1}H(k^{-1})^{2s-n} \sim \varrho_{2k} \leqslant c_1 e_{2k}(B_s) \sim e_{2k}(id^{\mu}) \\ \leqslant c_2 e_k^2(tr_{\mu}) \leqslant c_3 k^{-1} H(k^{-1})^{2s-n} \end{aligned}$$

resulting in (5.5). □

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